# ON THE WAVE EQUATION WITH QUADRATIC NONLINEARITIES IN THREE SPACE DIMENSIONS

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Abstract. The Cauchy problem for the nonlinear wave equation

$$\Box u = (\partial u)^2, \qquad u(0) = u_0,\ u_t(0) = u_1$$

in three space dimensions is considered. The data $(u_0, u_1)$ are assumed to belong to $\widehat{H}^r_s(\mathbb{R}^3) \times \widehat{H}^r_{s-1}(\mathbb{R}^3)$, where $\widehat{H}^r_s$ is defined by the norm

$$\|f\|_{\widehat{H}^r_s} := \|\langle\xi\rangle^s \widehat{f}\|_{L^{r'}_\xi}, \quad \langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}, \quad \frac{1}{r} + \frac{1}{r'} = 1.$$

Local well-posedness is shown in the parameter range $2 \ge r > 1$, $s > 1 + \frac{2}{r}$. For $r = 2$ this coincides with the result of Ponce and Sideris, which is optimal on the $H^s$-scale by Lindblad's counterexamples, but nonetheless leaves a gap of $\frac{1}{2}$ derivative to the scaling prediction. This gap is closed here except for the endpoint case. Corresponding results for $\Box u = \partial u^2$ are obtained, too.

## 1. Introduction and main results

In this note we consider the Cauchy problem

$$\Box u = (\partial_t^2 - \Delta) u = B_k(u,u), \qquad u(0) = u_0,\ u_t(0) = u_1 \tag{1}$$

for the nonlinear wave equation in $\mathbb{R}^3$, where the right hand side is given by

$$B_1(u,v) = \partial(uv) \qquad \text{or} \qquad B_2(u,v) = \partial u \partial v \tag{2}$$

with $\partial \in \{\partial_t, \partial_{x_1}, \partial_{x_2}, \partial_{x_3}\}$, and no special structure of the bilinear forms $B_k$, $k \in \{1,2\}$, such as a null-structure is assumed. Concerning the local well-posedness (LWP) of this problem with data $(u_0, u_1) \in H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$ the following is known. For $s > k + \frac{1}{2}$ energy estimates can be applied to obtain an affirmative result. Ponce and Sideris showed in [15] how to improve this down to $s > k$ by using Strichartz inequalities. Further progress is possible, if the nonlinearity satisfies a null-condition such as

$$\widetilde{B}_2(u,v) = \langle \nabla_x u, \nabla_x v\rangle - \partial_t u \partial_t v,$$

see the work of Klainerman and Machedon [8], [9], [10], who used wave Sobolev spaces to exploit the null-structure of the bilinear terms, thus reaching LWP for $s > s_c(k) = k - \frac{1}{2}$, which is here the critical Sobolev regularity by scaling considerations. If no such structure is present in the quadratic term, one has in fact ill-posedness of the Cauchy problem (1) for $s \le k$, as the sharp counterexamples of Lindblad show, see [12], [13], [14]. So in general there is a gap of half a derivative between the optimal LWP result on the $H^s$-scale and the scaling prediction.

For several important nonlinear dispersive equations in *one* space dimension - such as cubic NLS and DNLS, KdV, mKdV and its higher order generalizations -

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there is a similar gap between the best possible LWP result in $H^s$ and the critical regularity. In the case of cubic nonlinearities this can be closed almost completely by considering data in the spaces $\widehat{H}^r_s$, defined by the norm

$$\|f\|_{\widehat{H}^r_s} := \|\langle\xi\rangle^s \widehat{f}\|_{L^{r'}_\xi}, \quad \langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}, \quad \frac{1}{r}+\frac{1}{r'}=1,$$

see [3], [4], [7], [5]; for an application in the periodic setting cf. [6]. The purpose of this note is to show that the methods developed in the one-dimensional framework also apply to the nonlinear wave equation (1), (2) in three space dimensions and give LWP for data $(u_0,u_1) \in \widehat{H}^r_s(\mathbb{R}^3) \times \widehat{H}^r_{s-1}(\mathbb{R}^3)$, provided $1 < r \le 2$ and $s > s_k(r) := k+1-\frac{2}{r'}$. In the limit $r \to 1$ we can almost reach the space $\widehat{H}^1_{k+1}(\mathbb{R}^3) \times \widehat{H}^1_k(\mathbb{R}^3)$, which is critical by scaling. To prove this result we use an appropriate variant of Bourgain's Fourier restriction norm method, see [3, Section 2], and estimates for products of two free solutions of half-wave equations. The latter are very much in the spirit of the work of Foschi and Klainerman [1] and can be seen as bilinear substitutes and refinements of the Strichartz inequalities for the three-dimensional wave equation.

## 2. General arguments, function spaces, and precise statement of results

Following [2, Section 2] we first rewrite (1) as the first order system

$$(i\partial_t \mp J_x)u_\pm = \mp\frac{1}{4}J_x^{-1}B_k(u_+ + u_-) \mp \frac{1}{2}J_x^{-1}(u_+ + u_-), \tag{3}$$

where $J_x = (1-\Delta_x)^{\frac{1}{2}}$ is the Bessel potential operator of order $-1$ in the space variable $x$, and $u_\pm = u \pm iJ_x^{-1}\partial_t u$ so that the initial conditions become

$$u_\pm(0) = u_0 \pm iJ_x^{-1}u_1 =: f_\pm \in \widehat{H}^r_s(\mathbb{R}^3). \tag{4}$$

To treat the system (3) with data given by (4) we will use the function spaces $X^{r,\pm}_{s,b}$ defined by their norm

$$\|u\|_{X^{r,\pm}_{s,b}} := \left(\int d\xi d\tau \langle\xi\rangle^{sr'}\langle\tau \pm |\xi|\rangle^{br'}|\mathcal{F}u(\xi,\tau)|^{r'}\right)^{\frac{1}{r'}}.$$

For $s=b=0$ we write $\widehat{L^r_{xt}} := X^{r,+}_{0,0} = X^{r,-}_{0,0}$, correspondingly we set $\widehat{L^r_x} := \widehat{H}^r_0$. Local solutions are obtained by the contraction mapping principle in the time restriction space

$$X^{r,\pm}_{s,b}(\delta) := \{u = \tilde{u}|_{[-\delta,\delta]\times\mathbb{R}^3} : \tilde{u} \in X^{r,\pm}_{s,b}\}$$

endowed with the norm

$$\|u\|_{X^{r,\pm}_{s,b}(\delta)} := \inf\{\|\tilde{u}\|_{X^{r,\pm}_{s,b}} : \tilde{u}|_{[-\delta,\delta]\times\mathbb{R}^3} = u\}.$$

We will always have $b > \frac{1}{r}$, hence $X^{r,\pm}_{s,b} \subset C(\mathbb{R},\widehat{H}^r_s)$ and $X^{r,\pm}_{s,b}(\delta) \subset C([-\delta,\delta],\widehat{H}^r_s)$, which gives the persistence property of our solutions. To deal with $B_2$ - especially if time derivatives are involved - we also need the norms

$$\|u\|_{Z^{r,\pm}_{s,b}} := \|u\|_{X^{r,\pm}_{s,b}} + \|\partial_t u\|_{X^{r,\pm}_{s-1,b}};$$

the corresponding restriction spaces are defined precisely as above. Now our result concerning $B_1$ reads as follows.

**Theorem 1.** *Let $1 < r \le 2$, $s > \frac{2}{r}$, $\frac{1}{r} < b < 1$ and $f_\pm \in \widehat{H}^r_s$. Then there exist $\delta = \delta(\|f_+\|_{\widehat{H}^r_s}, \|f_-\|_{\widehat{H}^r_s}) > 0$ and a unique solution $(u_+,u_-) \in X^{r,+}_{s,b}(\delta) \times X^{r,-}_{s,b}(\delta)$ of*

(3) *with* $k = 1$ *satisfying the initial condition* (4)*. The solution is persistent and the flow map*

$$(f_+, f_-) \mapsto (u_+, u_-), \qquad \widehat{H}^r_s \times \widehat{H}^r_s \to X^{r,+}_{s,b}(\delta) \times X^{r,-}_{s,b}(\delta)$$

*is locally Lipschitz continuous.*

Similarly we can show the following for $B_2$.

**Theorem 2.** *Let* $1 < r \leq 2$, $s > \frac{2}{r} + 1$, $\frac{1}{r} < b < 1$ *and* $f_\pm \in \widehat{H}^r_s$*. Then there exist* $\delta = \delta(\|f_+\|_{\widehat{H}^r_s}, \|f_-\|_{\widehat{H}^r_s}) > 0$ *and a unique solution* $(u_+, u_-) \in Z^{r,+}_{s,b}(\delta) \times Z^{r,-}_{s,b}(\delta)$ *of* (3) *with* $k = 2$ *satisfying the initial condition* (4)*. The solution is persistent and the flow map*

$$(f_+, f_-) \mapsto (u_+, u_-), \qquad \widehat{H}^r_s \times \widehat{H}^r_s \to Z^{r,+}_{s,b}(\delta) \times Z^{r,-}_{s,b}(\delta)$$

*is locally Lipschitz continuous.*

The general LWP theorem [3, Theorem 2.3] reduces the proofs of Theorem 1 and 2 to that of bilinear estimates in $X^{r,\pm}_{s,b}$-norms. The next section is devoted to the proof of the following key estimate.

**Theorem 3.** *Let* $1 < r \leq 2$, $b > \frac{1}{r}$, *and* $\sigma > \frac{2}{r}$*. Then*

$$\|J^\sigma_x(uv)\|_{\widehat{L^r_{xt}}} + \|J^{\sigma-1}_x \partial_t(uv)\|_{\widehat{L^r_{xt}}} \lesssim \|u\|_{X^{r,\pm}_{\sigma,b}} \|v\|_{X^{r,[\pm]}_{\sigma,b}}, \tag{5}$$

*where* $[\pm]$ *denotes independent signs.*

Assume (5) already to be proven. Concerning $B_1$ we then have that for all $b, r$ and $s = \sigma$ according to the assumptions of Theorem 3 and $b' \leq 0$

$$\begin{aligned}\|J^{-1}_x \partial(uv)\|_{X^{r,\pm}_{s,b'}} &\leq \|J^{s-1}_x \partial(uv)\|_{\widehat{L^r_{xt}}} \\ &\leq \|J^s_x(uv)\|_{\widehat{L^r_{xt}}} + \|J^{s-1}_x \partial_t(uv)\|_{\widehat{L^r_{xt}}} \lesssim \|u\|_{X^{r,\pm}_{s,b}} \|v\|_{X^{r,[\pm]}_{s,b}},\end{aligned}$$

which combined with [3, Theorem 2.3] leads to Theorem 1, since the linear term on the right of (3) can be trivially taken care of. Similarly for $B_2$ we have with $s = \sigma + 1 > 1 + \frac{2}{r}$ and $r, b, b'$ as before

$$\begin{aligned}\|J^{-1}_x(\partial u \partial v)\|_{Z^{r,\pm}_{s,b'}} &\leq \|J^\sigma_x(\partial u \partial v)\|_{\widehat{L^r_{xt}}} + \|J^{\sigma-1}_x \partial_t(\partial u \partial v)\|_{\widehat{L^r_{xt}}} \\ &\lesssim \|\partial u\|_{X^{r,\pm}_{\sigma,b}} \|\partial v\|_{X^{r,[\pm]}_{\sigma,b}} \lesssim \|u\|_{Z^{r,\pm}_{s,b}} \|v\|_{Z^{r,[\pm]}_{s,b}},\end{aligned}$$

which is sufficient for Theorem 2.

## 3. Proof of the key estimate

Theorem 3 will be a consequence of several bilinear estimates for free solutions of the half-wave equations $(i\partial_t \pm D_x)u = 0$, subject to the initial condition $u(0) = u_0$. So for the remaining part of the paper let $u_\pm(t) = e^{\pm itD_x} u_0 = \mathcal{F}^{-1}_x e^{\pm it|\xi|} \mathcal{F}_x u_0$ and $v_\pm(t) = e^{\pm itD_x} v_0$. By the tranfer principle - see e. g. [11, Proposition 3.5] or [3, Lemma 2.1] - the proof of (5) essentially[1] reduces to showing that

$$\|J^{\sigma-1}_x \partial_x(u_\pm v_{[\pm]})\|_{\widehat{L^r_{xt}}} + \|J^{\sigma-1}_x \partial_t(u_\pm v_{[\pm]})\|_{\widehat{L^r_{xt}}} \lesssim \|u_0\|_{\widehat{H}^r_\sigma} \|v_0\|_{\widehat{H}^r_\sigma}. \tag{6}$$

[1]for low frequencies $|\xi| \leq 1$ we obtain $\|J^\sigma_x(uv)\|_{\widehat{L^r_{xt}}} \lesssim \|u\|_{X^{r,\pm}_{\sigma,b}} \|v\|_{X^{r,[\pm]}_{\sigma,b}}$ by Young's inequality and Sobolev type embeddings.

To prove (6) we make substantial use of the calculations in [1]. By symmetry it suffices to consider the $(++)$- and $(+-)$-cases. For both we calculate the space-time Fourier transform of the product. Defining $P_\pm(\eta) := |\frac{\xi}{2}-\eta| \pm |\frac{\xi}{2}+\eta|$ with $\nabla P_\pm(\eta) = \frac{\eta-\frac{\xi}{2}}{|\eta-\frac{\xi}{2}|} \pm \frac{\eta+\frac{\xi}{2}}{|\eta+\frac{\xi}{2}|}$ and using the properties of the $\delta$-distribution we obtain

$$\mathcal{F}u_+v_\pm(\xi,\tau) = c\int_{P_\pm(\eta)=\tau} \frac{dS_\eta}{|\nabla P_\pm(\eta)|}\widehat{u_0}(\frac{\xi}{2}-\eta)\widehat{v_0}(\frac{\xi}{2}+\eta),$$

for more details see [1, Sections 3 and 4]. Observe that the set $\{P_+(\eta)=\tau\}$ ($\{P_-(\eta)=\tau\}$) is an ellipsoid (hyperboloid) of rotation, so the $(++)$-case ($(+-)$-case) is henceforth referred to as elliptic (hyperbolic).

3.1. **The elliptic case.** We choose $0 < s_{1,2} < \frac{2}{r}$ with $s_1+s_2=\frac{2}{r}$ and use Hölder's inequality to get

$$|\mathcal{F}u_+v_+(\xi,\tau)| \lesssim \left(\int_{P_+(\eta)=\tau}\frac{dS_\eta}{|\nabla P_+(\eta)|}|\tfrac{\xi}{2}-\eta|^{-s_1r}|\tfrac{\xi}{2}+\eta|^{-s_2r}\right)^{\frac{1}{r}}\times$$
$$\left(\int_{P_+(\eta)=\tau}\frac{dS_\eta}{|\nabla P_+(\eta)|}|\widehat{J_x^{s_1}u_0}(\tfrac{\xi}{2}-\eta)\widehat{J_x^{s_2}v_0}(\tfrac{\xi}{2}+\eta)|^{r'}\right)^{\frac{1}{r'}}.$$

For the first factor we apply [1, Lemma 4.1] to see that

$$\int_{P_+(\eta)=\tau}\frac{dS_\eta}{|\nabla P_+(\eta)|}|\tfrac{\xi}{2}-\eta|^{-s_1r}|\tfrac{\xi}{2}+\eta|^{-s_2r}$$
$$= c\int_{-1}^{1}|\tau+|\xi|x|^{1-s_1r}|\tau-|\xi|x|^{1-s_2r}dx$$
$$= c\int_{-1}^{1}|\tfrac{\tau}{|\xi|}+x|^{1-s_1r}|\tfrac{\tau}{|\xi|}-x|^{1-s_2r}dx \le c_{s_1,s_2}.$$

Taking the $L^{r'}_{\xi,\tau}$-norm of the second factor and using the coarea formula we arrive at

$$\|u_+v_+\|_{\widehat{L^r_{xt}}} \lesssim \|J_x^{s_1}u_0\|_{\widehat{L^r_x}}\|J_x^{s_2}v_0\|_{\widehat{L^r_x}}.$$

Unfortunately this argument breaks down, if $s_1=0$ or $s_2=0$, cf. the necessary condition (9) in [1]. To overcome this difficulty we split $u_+v_+ = P_{\gtrsim}(u_+,v_+) + P_{\ll}(u_+,v_+)$, where

$$\mathcal{F}_xP_{\gtrsim}(f,g)(\xi) := \int_{\xi_1+\xi_2=\xi}\widehat{f}(\xi_1)\widehat{g}(\xi_2)\chi_{\{|\xi_1|\gtrsim|\xi_2|\}}d\xi_1.$$

By the preceeding we have

$$\|P_{\gtrsim}(u_+,v_+)\|_{\widehat{L^r_{xt}}} \lesssim \|J_x^{\frac{2}{r}}u_0\|_{\widehat{L^r_x}}\|v_0\|_{\widehat{L^r_x}}. \tag{7}$$

To estimate $P_{\ll}(u_+,v_+)$ we decompose $u_0$ dyadically into $u_0=\sum_{k\ge0}P_{\Delta k}u_0$ with $P_{\Delta 0}=\mathcal{F}_x^{-1}\chi_{\{|\xi|\le1\}}\mathcal{F}_x$ and, for $k\ge1$, $P_{\Delta k}=\mathcal{F}_x^{-1}\chi_{\{|\xi|\sim2^k\}}\mathcal{F}_x$, so that

$$\|P_{\ll}(u_+,v_+)\|_{\widehat{L^r_{xt}}} \le \sum_{k\ge0}\|P_{\ll}(P_{\Delta k}u_+,v_+)\|_{\widehat{L^r_{xt}}}.$$

Now by [1, Lemma 12.2] we have

$$\int_{P_\pm(\eta)=\tau}\frac{dS_\eta}{|\nabla P_\pm(\eta)|}\chi_{\{2^k\sim|\frac{\xi}{2}-\eta|\ll|\frac{\xi}{2}+\eta|\}} \lesssim 2^{2k}, \tag{8}$$

hence a Hölder application as above gives

$$\|P_{\ll}(P_{\Delta k}u_+,v_+)\|_{\widehat{L^r_{xt}}} \lesssim 2^{\frac{2k}{r}}\|P_{\Delta k}u_0\|_{\widehat{L^r_x}}\|v_0\|_{\widehat{L^r_x}}.$$

Summing up the dyadic pieces and combining the result with (7) we obtain for $\sigma > \frac{2}{r}$

$$\|u_+v_+\|_{\widehat{L^r_{xt}}} \lesssim \|J^\sigma_x u_0\|_{\widehat{L^r_x}} \|v_0\|_{\widehat{L^r_x}}. \tag{9}$$

The convolution constraint $\xi = \xi_1 + \xi_2 = (\frac{\xi}{2} - \eta) + (\frac{\xi}{2} + \eta)$ implies $\langle \xi \rangle^\sigma \lesssim \langle \xi_1 \rangle^\sigma + \langle \xi_2 \rangle^\sigma = \langle \frac{\xi}{2} - \eta \rangle^\sigma + \langle \frac{\xi}{2} + \eta \rangle^\sigma$, and we may exchange $u_0$ and $v_0$ in (9). This gives

$$\|J^\sigma_x(u_+v_+)\|_{\widehat{L^r_{xt}}} \lesssim \|J^\sigma_x u_0\|_{\widehat{L^r_x}} \|J^\sigma_x v_0\|_{\widehat{L^r_x}}, \tag{10}$$

provided $\sigma > \frac{2}{r}$. In (10) we may clearly replace $J^\sigma_x(u_+v_+)$ by $J^{\sigma-1}_x \partial_t(u_+v_+)$, since in the support of $\mathcal{F}(u_+v_+)$ we have $\tau = |\frac{\xi}{2} - \eta| + |\frac{\xi}{2} + \eta|$ and hence $\langle \xi \rangle^{\sigma-1}|\tau| \le \langle \frac{\xi}{2} - \eta \rangle^\sigma + \langle \frac{\xi}{2} + \eta \rangle^\sigma$. Thus we have shown:

**Lemma 1.** *Let $1 \le r \le 2$ and $\sigma > \frac{2}{r}$. Then*

$$\|J^\sigma_x(u_+v_+)\|_{\widehat{L^r_{xt}}} + \|J^{\sigma-1}_x \partial_t(u_+v_+)\|_{\widehat{L^r_{xt}}} \lesssim \|u_0\|_{\widehat{H}^r_\sigma} \|v_0\|_{\widehat{H}^r_\sigma}.$$

3.2. **The hyperbolic case.** The estimation in this case goes along similar lines as in section 3.1, as long as

$$|\frac{\xi}{2} - \eta| + |\frac{\xi}{2} + \eta| \le c_1|\xi|. \tag{11}$$

If (11) is fulfilled, we choose again $s_{1,2} \in (0, \frac{2}{r})$ with $s_1 + s_2 = \frac{2}{r}$ and obtain from [1, Lemma 4.4] that

$$\int_{P_-(\eta)=\tau} \frac{dS_\eta}{|\nabla P_-(\eta)|} |\tfrac{\xi}{2} - \eta|^{-s_1 r} |\tfrac{\xi}{2} + \eta|^{-s_2 r} \chi_{\{(11)\}}$$
$$= c \int_1^{c_1} |\tfrac{\tau}{|\xi|} + x|^{1-s_1 r} |\tfrac{\tau}{|\xi|} - x|^{1-s_2 r} dx \le c_{s_1,s_2},$$

which gives

$$\|u_+v_-\|_{\widehat{L^r_{xt}}} \lesssim \|J^{s_1}_x u_0\|_{\widehat{L^r_x}} \|J^{s_2}_x v_0\|_{\widehat{L^r_x}}$$

and hence

$$\|P_{\gtrsim}(u_+, v_-)\|_{\widehat{L^r_{xt}}} \lesssim \|J^{\frac{2}{r}}_x u_0\|_{\widehat{L^r_x}} \|v_0\|_{\widehat{L^r_x}}. \tag{12}$$

A dyadic decomposition together with (8) shows that

$$\|P_{\ll}(P_{\Delta k} u_+, v_-)\|_{\widehat{L^r_{xt}}} \lesssim 2^{\frac{2k}{r}} \|P_{\Delta k} u_0\|_{\widehat{L^r_x}} \|v_0\|_{\widehat{L^r_x}}, \tag{13}$$

and combining (12) and (13) after summation in $k$ we arrive at

$$\|u_+v_-\|_{\widehat{L^r_{xt}}} \lesssim \|J^\sigma_x u_0\|_{\widehat{L^r_x}} \|v_0\|_{\widehat{L^r_x}},$$

provided $1 \le r \le 2$, $\sigma > \frac{2}{r}$, and $u_+v_-$ fulfills assumption (11). To fix a partial result concerning the hyperbolic case, let $P(u, v)$ denote the projection on the domain in Fourier space, where (11) holds. Then, taking into account the arguments at the end of Section 3.1, we have the following estimate.

**Lemma 2.** *Let $1 \le r \le 2$ and $\sigma > \frac{2}{r}$. Then*

$$\|J^\sigma_x P(u_+, v_-)\|_{\widehat{L^r_{xt}}} + \|J^{\sigma-1}_x \partial_t P(u_+, v_-)\|_{\widehat{L^r_{xt}}} \lesssim \|u_0\|_{\widehat{H}^r_\sigma} \|v_0\|_{\widehat{H}^r_\sigma}.$$

We turn to the region, where

$$c_1|\xi| < |\frac{\xi}{2} - \eta| + |\frac{\xi}{2} + \eta|, \tag{14}$$

and denote the projection thereon by $Q(u,v)$. We apply again [1, Lemma 4.4] with $F(|\frac{\xi}{2}-\eta|,|\frac{\xi}{2}+\eta|)=|\frac{\xi}{2}-\eta|^{-s_1r}|\frac{\xi}{2}+\eta|^{-s_2r}\chi_{\{(14)\}}$, where $s_{1,2}\geq 0$ and $s_1+s_2=\frac{3}{r}+\varepsilon$. This gives

$$\int_{P_-(\eta)=\tau}\frac{dS_\eta}{|\nabla P_-(\eta)|}|\tfrac{\xi}{2}-\eta|^{-s_1r}|\tfrac{\xi}{2}+\eta|^{-s_2r}\chi_{\{(14)\}}$$
$$=c\int_{c_1}^{\infty}|\tau+|\xi|x|^{1-s_1r}|\tau-|\xi|x|^{1-s_2r}dx$$
$$=c|\xi|^{2-(s_1+s_2)r}\int_{c_1}^{\infty}|\tfrac{\tau}{|\xi|}+x|^{1-s_1r}|\tfrac{\tau}{|\xi|}-x|^{1-s_2r}dx\lesssim|\xi|^{2-(s_1+s_2)r},$$

which in turn implies

$$\|D_x^{s_1+s_2-\frac{2}{r}}Q(u_+,v_-)\|_{\widehat{L^r_{xt}}}\lesssim\|J_x^{s_1}u_0\|_{\widehat{L^r_x}}\|J_x^{s_2}v_0\|_{\widehat{L^r_x}}. \tag{15}$$

Bilinear interpolation of (15) with $r=1$ and

$$\|u_+v_-\|_{L^2_{xt}}\lesssim\|J_x^{\sigma_1}u_0\|_{L^2_x}\|J_x^{\sigma_2}v_0\|_{L^2_x},\qquad(\sigma_{1,2}\geq 0,\sigma_1+\sigma_2>1),$$

which follows from Strichartz estimate, gives the sharpened version

$$\|D_x^sQ(u_+,v_-)\|_{\widehat{L^r_{xt}}}\lesssim\|J_x^{s_1}u_0\|_{\widehat{L^r_x}}\|J_x^{s_2}v_0\|_{\widehat{L^r_x}},$$

where $1\leq r\leq 2$, $s=(1-\frac{2}{r'})(1+\varepsilon)$, $s_{1,2}\geq 0$ with $s_1+s_2=3-\frac{4}{r'}+\varepsilon$ and $\varepsilon>0$. If in addition $r>1$ and $\varepsilon$ is sufficiently small, so that $s\leq 1$, we may replace the $D_x^s$ by $J_x^{s-1}\partial_x$ and hence by $J_x^{-\frac{2}{r'}}\partial_x$. This gives

$$\|J_x^{-\frac{2}{r'}}\partial_xQ(u_+,v_-)\|_{\widehat{L^r_{xt}}}\lesssim\|J_x^{s_1}u_0\|_{\widehat{L^r_x}}\|J_x^{s_2}v_0\|_{\widehat{L^r_x}}$$

for all $r\in(1,2]$ and $s_{1,2}\geq 0$ with $s_1+s_2>3-\frac{4}{r'}$. Using once more $\langle\xi\rangle\leq\langle\frac{\xi}{2}-\eta\rangle+\langle\frac{\xi}{2}+\eta\rangle$ we conclude for $\sigma>\frac{2}{r}$ that

$$\|J_x^{\sigma-1}\partial_xQ(u_+,v_-)\|_{\widehat{L^r_{xt}}}\lesssim\|u_0\|_{\widehat{H}^r_\sigma}\|v_0\|_{\widehat{H}^r_\sigma},$$

which also holds true with $\partial_t$ instead of $\partial_x$, since we are in the hyperbolic case, where $|\tau|\leq|\xi|$. Summarizing we have:

**Lemma 3.** *Let $1<r\leq 2$ and $\sigma>\frac{2}{r}$. Then*

$$\|J_x^{\sigma-1}\partial_xQ(u_+,v_-)\|_{\widehat{L^r_{xt}}}+\|J_x^{\sigma-1}\partial_tQ(u_+,v_-)\|_{\widehat{L^r_{xt}}}\lesssim\|u_0\|_{\widehat{H}^r_\sigma}\|v_0\|_{\widehat{H}^r_\sigma}.$$

Now the crucial estimate (6) follows from the Lemmas 1, 2, and 3.

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